DOES EXPOSURE TO HIGHER LEVEL MATHEMATICS AFFECT PROBLEM SOLVING?

John Kennedy
Bland Independent School District

Jackie Vogel
Austin Peay State University

Abstract

Does exposure to higher level mathematics affect problem solving, and therefore, critical thinking? To answer this question, a group of students from a small rural high school were presented with a constructivist problem that contained multiple entry points and multiple solution pathways. The students were asked to solve the problem in as many ways as possible and then justify or prove their answer to someone else. The problem posed to the students was how many unique towers could be constructed from two different colors of cubes and the towers were four cubes tall. The students were enrolled in Algebra 1, statistics, and pre-calculus. Results were evaluated in accordance with a rubric. The results indicated that exposure to higher level mathematics did not necessarily correlate into more sophisticated solution methods.

Keywords: problem solving, critical thinking, mathematical justification and informal

---

As part of the researcher's master's program, the researcher watched many videos that documented Dr. Maher's longitudinal study of problem solving and justification. She followed a group of children from second grade through twelfth and would occasionally present them with various mathematical challenges. One of the problems presented in the second grade was the tower problem. The tower problem involved determining how many towers can be built that were four units high, given two colors. She focused on asking them how they knew they had the answer. She was asking them to prove their conclusions in an informal way (Maher, 2011). The researcher thought that this would be a good problem to present to my students about needing to prove or convince somebody else of your conclusion or position. We do this all the time in real life, whether it is asking for a pay raise or spouses deciding on a restaurant. Trial lawyers make a living convincing others that what they say is true. The foundation for justifying one's position is laid in mathematical proofs. The *State of Texas Mathematical Process Standards* for high school students states that students are expected to "display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication" (19 TAC, 2012, para. 111.41).

It would stand to reason that as children are exposed to increasingly more sophisticated mathematics, both conceptually and mechanically, then the types of solutions and their justifications for those solutions should also become more sophisticated. Does higher level mathematics lead to higher level thinking and, therefore, more complex solutions? Or, on the other hand, will students, and people in general, opt for a simpler solution that just solves the problem at hand? The expected outcomes are that Algebra 1 students will produce solutions that are less complex than students enrolled in statistics or pre-calculus. It is also expected that some variance will occur between the statistics group and the pre-calculus group.

The primary research question is: Does exposure to higher level mathematics affect problem solving?

The secondary research question is: In what ways does exposure to different levels of mathematics affect students' critical thinking when presented with a problem involving multiple entry points?

ISSN: 2166-0190 online

Literature Review

Critical Thinking

In the current climate of preparing students to be college and career ready, a significant mandate under state and federal law is to elevate the students' level of problem solving and critical thinking. It is a specific objective under the Public Law 107-110, commonly referred to as the No Child Left Behind Act (107th Congress, 2001) and its successor, Every Student Succeeds Act of 2015 ("114th Congress", 2015). On the state and local level, 42 states have adopted the Common Core State Standards ("Standards in your State", 2016) which also strongly emphasize critical thinking ("P21 Toolkit," 2011). Although Texas has not adopted Common Core, its own state standards also contain elements that are specifically directed at critical thinking ("House Bill 5", 2014). According to Scriven and Paul (n.d.), "critical thinking" is the process of analyzing, synthesizing, and/or evaluating information and entails the examination of reasoning leading to conclusions.

Problem Solving

The concept of problem solving has changed over the years. Decades ago grade school and high school students were taught to use formulas to solve various questions. If students could do this, they were good at math. Word problems were usually given as an enrichment exercise (Lester, 1994). The problem was that you could be a good test-taker and still have low critical thinking skills. In the 1980s and 1990s, the role of metacognition came to the forefront and word problems involving a real world or situation context became the norm (Cai & Lester, 2010). The focus has shifted to presenting students with problems that have multiple entry points with multiple solution pathways (NCTM, 1989). Furthermore, students can better understand theorems by participating in their discovery and proof (Nord, Malm, & Nord, 2002).

The question then becomes what is a good or worthwhile problem? According to the National Council of Teachers of Mathematics (1991), a worthwhile mathematical question directs students to investigate important mathematical ideas and ways of thinking, usually towards a learning objective. Cai and Lester (2010, p. 5) set forth a list of ten criteria for worthwhile problems. Their criteria are as follows:

1) The problem has important, useful mathematics embedded in it.
2) The problem requires higher-level thinking and problem solving.
3) The problem contributes to the conceptual development of students.
4) The problem creates an opportunity for the teacher to assess what the students are learning and where they are having trouble.
5) The problem can be approached by students in multiple ways using different solution strategies.
6) The problem has various solutions or allows different decisions or positions to be taken or defended.
7) The problem encourages student engagement and discourse.
8) The problem connects to other important mathematical ideas.
9) The problem promotes the skillful use of mathematics.
10) The problem provides an opportunity to practice important skills.

Of course, it is unreasonable to expect every question to meet all ten criteria. Earlier, Smith and Stein (1998) developed a list of characteristics describing four levels of cognitive demand. Those levels of cognitive demand are lower-level demands (memorization), lower-level demands (procedures without connections), higher-level demands (procedures with connections), and higher-level demands (doing mathematics). The first two levels (memorization and procedures without connections) correlate to the lower levels of Bloom's Taxonomy and involve the memorization and use of formulas.

An example would be to find the intersection of two given linear equations. The third category (procedures with connections) involves the decoding of word problems. The last category (doing mathematics) deserves further elaboration. According to Smith and Stein (1998, p. 348), the characteristics of doing mathematics are as follows:

- Require complex and nonalgorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.

ISSN: 2166-0190 online

- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solution.
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

According to Mueller, Yankelewitz, and Maher (2014), meaningful mathematical learning involves selecting challenging, open-ended tasks that require students to extend their learning and justify their solutions.

Teaching Strategies

Teaching strategies have changed over the years from rote memorization of formulae with purely mechanical drill application to modern constructivism in which the student is given the freedom to formulate their own conclusions and only those conclusions may therefore be considered valid (Schoenfeld, 1992). These two extremes can be exemplified using an economic analogy. The former strategy declares there is only a single way to solve a problem. Even though other processes may lead to the same conclusion, if the prescribed method is not used, the conclusion is invalid. This is akin to a rigorously controlled centralist economy such as North Korea. The latter strategy postulates that the only valid conclusions are the ones the students themselves make. This is reminiscent of the laisse-faire economic model of Adam Smith and others that proposed no government (i.e. teacher) intervention was needed and an "invisible hand" would lead the market to regulate itself for the benefit of all. Neither of these strategies, on its own, is practical. The first stifles independent thought and the second is open to false conclusions based on invalid assumptions. The optimal strategy is somewhere in-between.

The current high school curriculum often does not allow for discovery learning tasks designed to allow students to formulate and develop valid conclusions. Boaler (2016) and Flynn (2017) both advocate for tasks that have many entry points, multiple solution pathways, and multiple representations. Seeley (2017) advocates for an "upside-down" teaching model that is teacher-structured but centered on student thinking.

Best practices state that the best learning environment is student-centered, but teacher led (Dougherty & Rutherford, 2009). Student-centered learning leads students to have awareness that they are responsible for their own learning and gives them responsibility for their conclusions (Maher, 2002).

Methodology

Purpose and Research Questions

The purpose of this project was to investigate whether exposure to higher level mathematics affect the complexity of students' problem-solving skills and does exposure to different levels of mathematics affect a students' critical thinking when presented with a problem involving multiple entry points?

Setting and Participants

Potential participants were approximately 80 high school students. These individuals were between 15-18 years of age and in generally good health. Approximately 53% were males and 47% were females. Hispanic students comprised 40% of the population, white students were 55%, with the remaining 5% made up of African American students or people of two or more races. Approximately 25% were considered "at risk" and 53% were classified as economically disadvantaged. Target groups were students enrolled in Algebra 1, statistics and pre-calculus. Contact was made by the high school counselor during a normal school day during normal class hours. It was made clear that this event will be a collaborative learning opportunity and that the results will be used for this study. Anyone whose parents have declined to give consent to

ISSN: 2166-0190 online

the research aspect of the project will still participate in the problem-solving exercise as part of regular class activities, but their data set will not be included in the analysis.

Procedures

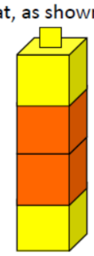
The researcher focused on one aspect of critical thinking (the process of analyzing, synthesizing, and/or evaluating information) and observed the types of reasoning leading to a conclusion. The observed students were divided into three distinct groups: those enrolled in Algebra 1, those enrolled in statistics, and those enrolled in pre-calculus. Working in small groups, all students were given the same open-ended question using a physical model that has a simple answer and asked to formulate a justification or proof that would convince someone else that their answer was correct. The exercise is modeled on one used by Dr. Maher during her Longitudinal Study (Maher, 2011). They were also asked to extrapolate or extend their solution to a similar problem that cannot be physically modeled. Justifications were categorized by their complexity in accordance with a rubric (Table 1).

Students were divided into groups of 2-3 and given an instruction sheet and 40 Unifix cubes of one color and 40 cubes of a different color (Figure 1).

The Towers Problem

Towers

How many different towers four cubes tall can be constructed when you have two colors of cubes from which to choose? The tower is built from Unifix cubes and should be able to sit flat, as shown.



Is there a way to predict the number of possible different towers, given the height of a tower and choosing cubes from two colors?

How would you convince or prove to someone else that you know you have the correct answer? How many different ways can you think of to do this?

Figure 1. The Towers Problem

They were asked to find how many different towers four cubes tall could be created and how they would justify or prove their conclusion to someone else. They were asked to record their solution(s) on the instruction sheet. Three 45-minute class periods were allotted for this activity. The completed Towers worksheet was collected and used to analyze the complexity of the solutions. The following rubric was used.

ISSN: 2166-0190 online

Table 1
Analysis Rubric

Level of Complexity	Description
Simple	Answer is only the correct numerical value, or the justification only uses the concept of doubling. Lacks mathematical formulation or evidence of logical deduction.
Moderate	Answer is the correct numerical value justified by a binary formula (2^4) or logical deduction in the form of a tree diagram or case analysis.
Intermediate	Answer is the correct numerical value justified by using Pascal's Triangle, a summation of combinatorics, or demonstrates evidence of abstraction from the concrete model.
Advanced	Answer is the correct numerical value justified by a full binomial expansion, $(x+y)^4$, or demonstrates some process to mathematically determine the composition of a tower or group of towers by abstraction of the problem, number of cubes per color. Demonstrates abstraction of the problem.

The number of responses per category for both groups was tallied, converted to a percent output, and then represented graphically. The relative distributions of each answer category were then compared between the three groups (Algebra 1, statistics, and pre-calculus) to determine if the students exposed to higher level mathematics demonstrate more complex problem-solving methodologies and, therefore, higher levels of critical thinking.

Qualitative Methods

The primary research question is: Does exposure to higher level mathematics affect problem solving?

The secondary research question is: In what ways does exposure to different levels of mathematics affect a students' critical thinking when presented with a problem involving multiple entry points?

To answer these questions, the students' written responses were analyzed for emerging themes. Common themes were coded using different colored highlighters as they were related to the primary and secondary research questions.

Quantitative Methods

The sophistication of student responses was analyzed in accordance with the above rubric, percentages were calculated by category, and relative frequencies were represented graphically.

Results

There were 77 students that returned the required consent and assent forms. They were divided into 37 groups of two to three students that resulted in n=37 data points. The output was sorted by class type (algebra, statistics and pre-calculus) and rubric category. During the sorting process, it was discovered that some students did not generate the correct answer. This outcome was not anticipated at the outset of the project. A fifth category, Incorrect, was added to the rubric to account for these results. The new rubric is shown in Table 2 below.

ISSN: 2166-0190 online

Table 2
Revised Analysis Rubric 2

Level of Complexity	Description
Incorrect	The correct numerical answer is not provided or there is a gross logical error.
Simple	Answer is only the correct numerical value, or the justification only uses the concept of doubling. Lacks mathematical formulation or evidence of logical deduction.
Moderate	Answer is the correct numerical value justified by a binary formula (2^4) or logical deduction in the form of a tree diagram or case analysis.
Intermediate	Answer is the correct numerical value justified by using Pascal's Triangle, a summation of combinatorics, or demonstrates evidence of abstraction from the concrete model.
Advanced	Answer is the correct numerical value justified by a full binomial expansion, $(x+y)^4$, or demonstrates some process to mathematically determine the composition of a tower or group of towers by abstraction of the problem, number of cubes per color. Demonstrates abstraction of the problem.

The results of the students' answer complexity by class category are summarized in Table 3.

Table 3
Answer Complexity Results Matrix

Class	Incorrect	Simple	Moderate	Intermediate	Advanced	Total
Algebra 1	2	13	4	2	0	21
Statistics	0	2	3	2	1	8
Pre-calculus	5	2	1	0	0	8
Total	7	17	8	4	1	37

The following graphical representations of the data are provided for the reader. Figure 2 represents the percentage Incorrect by class type. The percentages are as follows: Algebra 1≈29%, Statistics≈0%, Pre-calculus≈71%. An example of a commonly reported incorrect response would be 32.

ISSN: 2166-0190 online

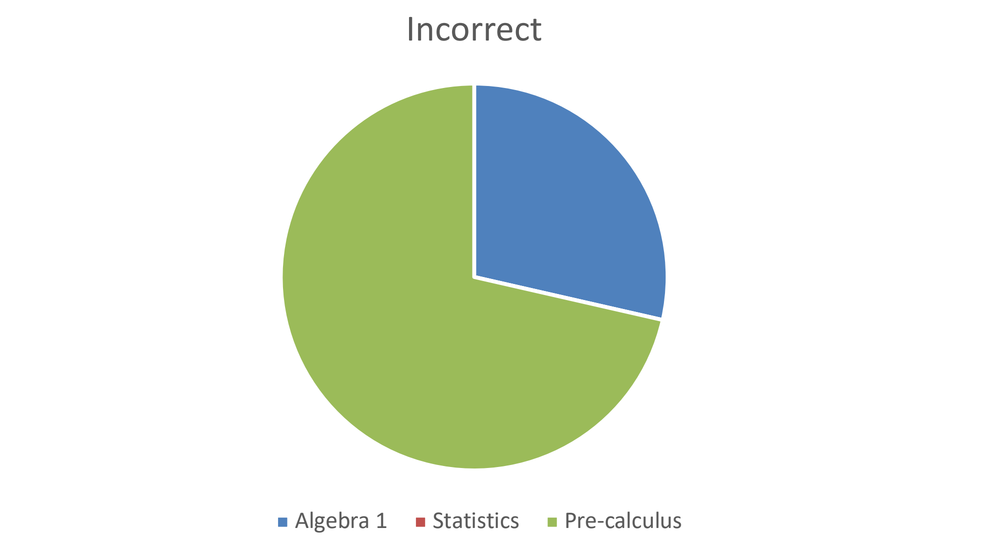


Figure 2. Incorrect Category by Class Type

Figure 3 represents the percentage of responses classified as Simple by class type. The percentages are as follows: Algebra 1≈76%, Statistics≈12%, Pre-calculus≈12%. An example of a simple response would be 16 without any explanation.

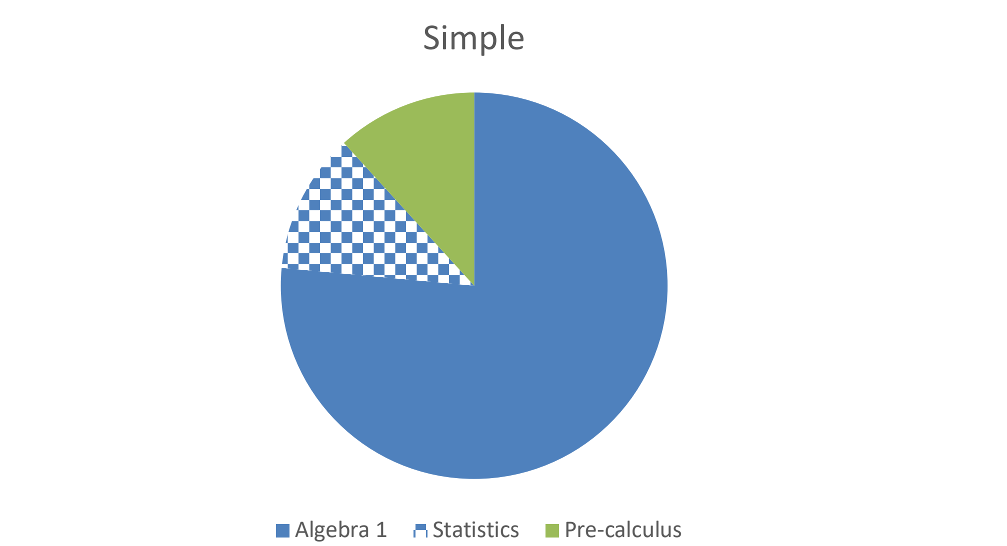


Figure 3. Simple Category by Class Type

ISSN: 2166-0190 online

Figure 4 represents the percentage of responses classified as Moderate by class type. The percentages are as follows: Algebra 1≈50%, Statistics≈38%, Pre-calculus≈13%. An example of a moderate response would be analyzing all of the possible cases to arrive at the answer of 16.

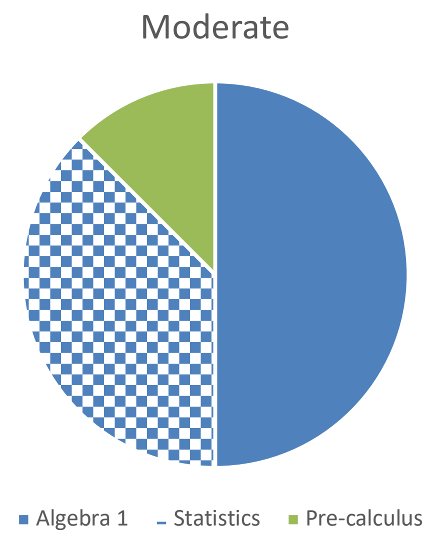


Figure 4. Moderate Category by Class Type

Figure 5 represents the percentage of responses classified as Intermediate by class type. The percentages are as follows: Algebra 1≈50%, Statistics≈50%, Pre-calculus≈0%. An example of an intermediate response would be using Pascal's Triangle to arrive at the answer of 16.

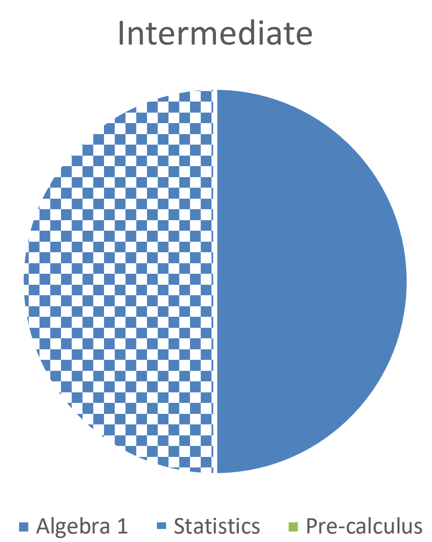


Figure 5. Intermediate Category by Class Type

ISSN: 2166-0190 online

Figure 6 represents the percentage of responses classified as Advanced by class type. The percentages are as follows: Algebra 1≈0%, Statistics≈100%, Pre-calculus≈0%. An example of an advanced response would be to use the binomial expansion to justify the conclusion of 16.

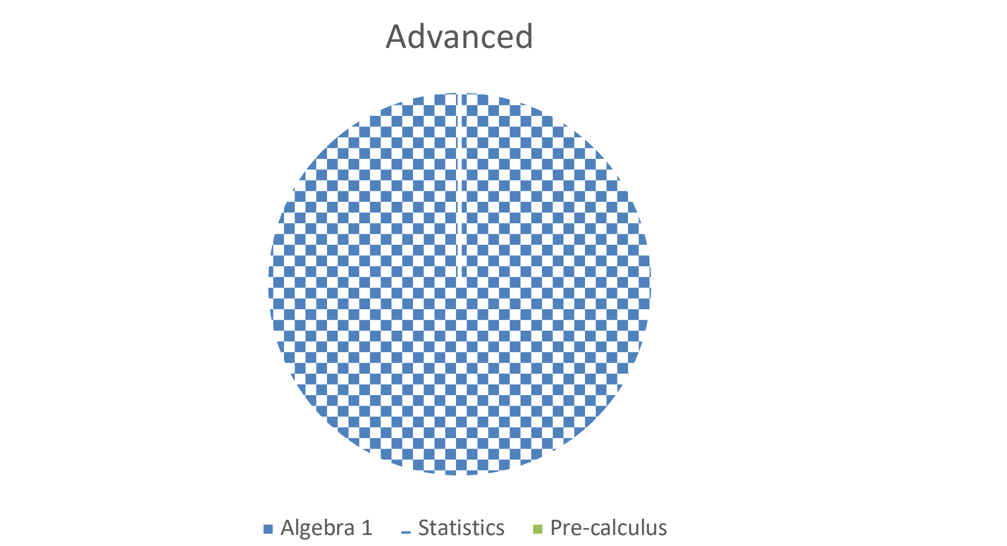


Figure 6. Advanced Category by Class Type

Figure 7 shows the category distribution for Algebra 1.

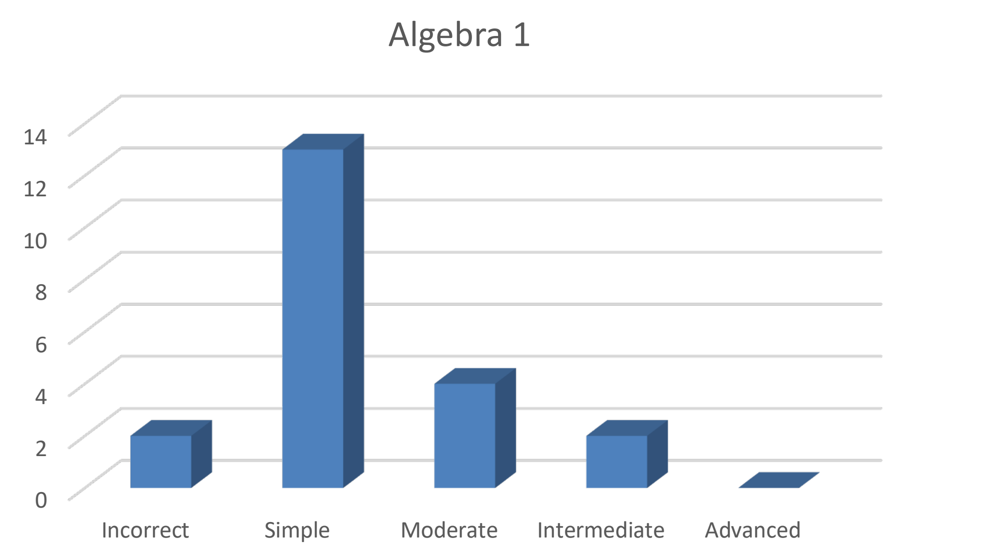


Figure 7. Category Frequency Distribution for Algebra 1

ISSN: 2166-0190 online

Figure 8 shows the category distribution for Statistics.

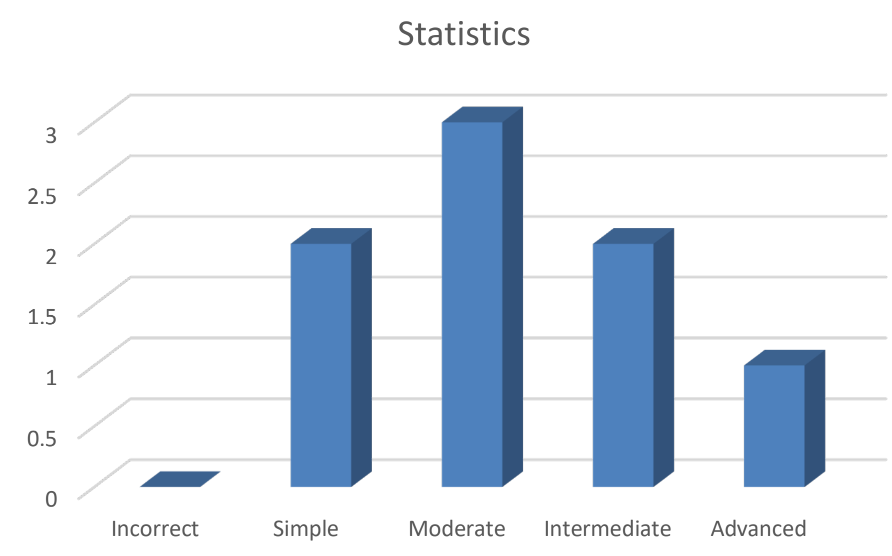


Figure 8. Category Frequency Distribution for Statistics

Figure 9 shows the category distribution for Pre-calculus.

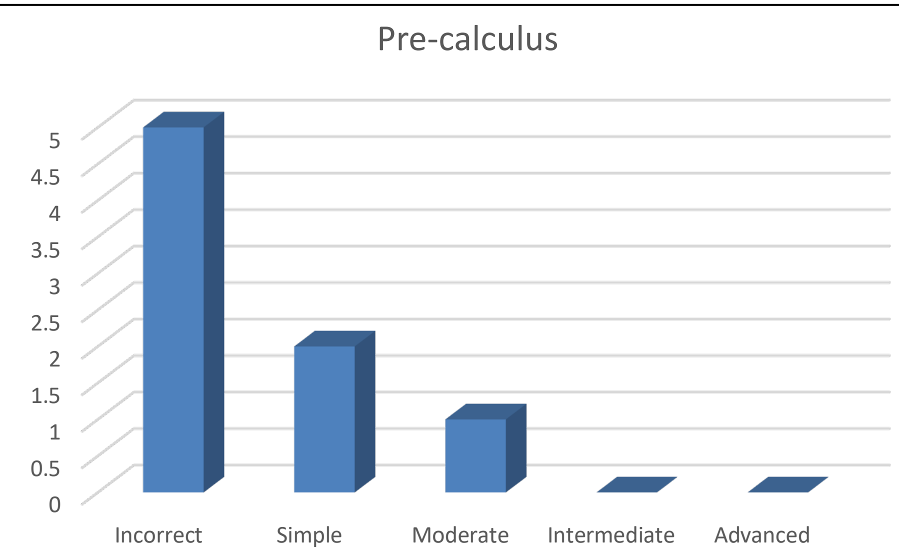


Figure 9. Category Frequency Distribution for Pre-calculus

ISSN: 2166-0190 online

Figure 10 shows the frequency distribution of students exposed to lower level mathematics (Algebra 1) compared to those exposed to higher level mathematics (statistics and pre-calculus combined).

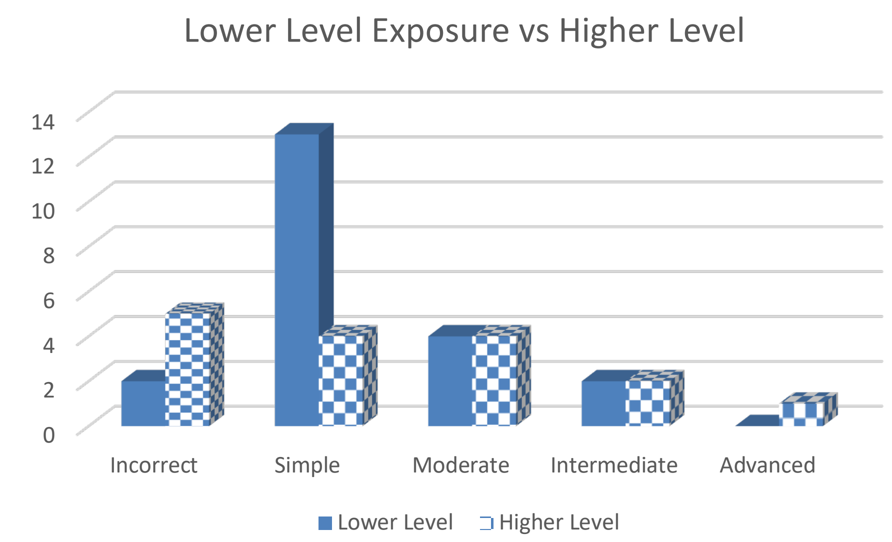


Figure 10. Lower Level vs Higher Level Mathematics Exposure

Observed Patterns and Themes

The following patterns and themes were observed by the researcher:

- There was a need to modify the original rubric to include a category for incorrect answers.
- Of the seven samples classified as incorrect, only one attempted a tree diagram and only one attempted a concrete model.
- The most common incorrect answer was 32 (4 of 7).
- 41% of the responses that were classified as simple gave a generalization as 4^2 which is the correct answer, 16, but demonstrates incorrect logic.
- Approximately 59% of the algebra students used some form of concrete modeling.
- Only the statistics students used any form of combinatorics.
- When comparing lower level versus higher level mathematics exposure, the number of responses in the top three categories is approximately equivalent.

Discussion

Analysis

It would stand to reason that as children are exposed to increasingly more sophisticated mathematics, both conceptually and mechanically, then the types of solutions and their justifications for those solutions should also become more sophisticated.

As a reminder, the primary research question is: Does exposure to higher level mathematics affect problem solving? The expectation was that students that had a lower level of exposure to mathematics would have more responses in the Simple and Moderate categories and students with more exposure would have more responses in the Intermediate and Advanced categories. Although the former was true, the latter was not. There were, in fact, a higher percentage of Incorrect responses for the higher-level group than the lower level one. Part of the reason may be that the students in the higher-level

ISSN: 2166-0190 online

group attempted to develop a purely abstract model from the outset which is evidenced by the lack of concrete modeling by those students and a common assertion that the answer was 32, exactly twice as big as it should be.

Among the three higher categories (Moderate, Intermediate, and Advanced) a Chi-Square analysis could not be completed due to not meeting the required test assumptions; however, there appears to be no meaningful difference between the two groups. The results do not support the conjecture that exposure to higher level mathematics leads to more complex solutions indicative of higher levels of critical thinking.

The secondary research question is: In what ways does exposure to different levels of mathematics affect students' critical thinking when presented with a problem involving multiple entry points? Students must be trained from an early age to learn how to solve problems with multiple entry points and multiple solution pathways. At the beginning of this project, it never occurred to the researcher that some students would not be able to solve the problem at a minimal level given that the students had manipulatives to aid in visualization of the problem. It is thought that without a clearly defined problem type or entry point, students had difficulty in developing an effective problem-solving strategy. For students, overall, to be capable and competent with constructivist problems, it cannot be an occasional experience but must be a long-term process imbedded as part of a comprehensive curriculum.

Limitations

One limitation was the small sample size. This project was conducted in a small rural high school in a Title 1 district in north Texas. The average graduating class is approximately 40 students. In an environment with a larger population, the results would probably be significantly different.

A second limitation, which is directly related to the small school population, is that there are only two mathematics teachers on the high school staff. This limits the students' experience regarding teaching styles. This lack of instructional diversity may account for some of the observed results, particularly between the statistics and pre-calculus groups.

Implications for Further Study

A more accurate conclusion might be obtained by obtaining a larger sample size, either from a larger school or across a series of schools. This would also negate the lack of instructional diversity mentioned above.

Although not part of this study, it may be interesting to track and account for students that are identified as special education (SPED) and/or gifted and talented (GT). These students were part of this project, but the data could not be disaggregated into SPED and GT. It would have been interesting to note how the results would have been affected with this additional level of disaggregated data.

Conclusions

The results of this study have implications for both preservice and in-service teachers. Frequently, preservice teachers, and some in-service teachers, believe that open-ended and problem-solving tasks are too complicated or time consuming to be of value and take time away from the daily standard. However, tasks of this nature reveal unexpected answers, both correct and incorrect, lead to a deeper understanding of how students think and construct mathematical arguments and address multiple standards in a single lesson.

ISSN: 2166-0190 online

References

19 TAC Chapter 111. Texas Essential Knowledge and Skills for Mathematics. (n.d.). Retrieved from http://ritter.tea.state.tx.us/rules/tac/chapter111/index.html

107th Congress (2001). Public law 107-110. *Congressional Record*, *147*, 146. Retrieved from http://www2.ed.gov/policy/elsec/leg/esea02/107-110.pdf

114th Congress (2015). Public law 114-95, Every student succeeds act. *Congressional Record*, *161*, 136. Retrieved from https://www.congress.gov/public-laws/114th-congress

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages, and innovative teaching*. San Francisco, CA: Josey-Bass.

Cai, J., & Lester, F. (2010). Why is teaching with problem solving important to student learning? *Research Brief*, 1-6. Retrieved from www.nctm.org

Dougherty, C. & Rutherford, J. (2009). The NCEA core practice framework: an organizational guide to sustained school improvement. Retrieved from http://www.act.org/content/dam/act/unsecured/documents/NCEA-Organizing-Guide.pdf

Flynn, M. (2017). From answer-getters to problem solvers. *Educational Leadership*, *75*(2), 26-31.

House Bill 5 and high school graduation requirements. (2014). Retrieved from http://www.tasanet.org/cms/lib07/TX01923126/Centricity/Domain/175/external/tea-ppt.pdf

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education*, *25*, 660-675. Retrieved from www.researchgate.net/publication/308645660

Maher, C. (2002). How students structure their own investigations and educate us: what we've learned from a fourteen year study. Retrieved from www.researchgate.net/publication/234571159

Maher, C. (2011). The longitudinal study. Retrieved from https://www.researchgate.net/publication/225960437

Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. *Investigations in Mathematics Learning*, *7*(2), 1-20.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA

National Council of Teachers of Mathematics (NCTM) (1991). *Professional standards for teaching mathematics*. Reston, VA

No Child Left Behind Act of 2001. (2001).

Nord, G., Malm, E. J., & Nord, J. (2002). Counting pizzas: a discovery lesson using combinatorics. *Mathematics Teacher*, *95*(1), 8-14. Retrieved from www.nctm.org

P21 Common core toolkit. (2011). Retrieved from www.p21.org

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning*, 334-370. New York: MacMillan.

Scriven, M., & Paul, R. (n.d.). *Defining critical thinking*. Retrieved from http://www.criticalthinking.org/pages/defining-critical-thinking/410

Seeley, C. (2017). Turning teaching upside down. *Educational Leadership*, *75*(2), 32-36.

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: from research to practice. *Mathematics Teaching in the Middle School*, *3*(5), 344-350.

Standards in your state. (2016). Retrieved from http://www.corestandards.org/standards-in-your-state/

Volume 8, pp. 59-71
ISSN: 2166-0190 online